QUADRATIC HYPERBOLICITY PRESERVERS & MULTIPLIER SEQUENCES

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Abstract. It is known (see [4, Brändén, Lemma 2.7]) that a necessary condition for $T := \sum Q_k(x)D^k$ to be hyperbolicity preserving is that $Q_k(x)$ and $Q_{k-1}(x)$ have interlacing zeros. We characterize all quadratic linear operators, as a consequence we find several classes of P_n-multiplier sequence.

1. Introduction

It is well known (see [8],[9, p. 32]) that if T is any linear operator defined on the space of real polynomials, $R[x]$, then there is a sequence of real polynomials, $\{Q_k(x)\}$, such that

$$T = \sum Q_k(x)D^k, \text{ where } D = \frac{d}{dx}. \tag{1.1}$$

Our investigation involves such operators that act on polynomials, in particular, we are interested in polynomials with the following property.

Definition 1. A polynomial $f(x) \in \mathbb{R}[x]$ whose zeros are all real is said to be *hyperbolic*. Following the convention of G. Pólya and J. Schur [10, p.89], the constant 0 is also deemed to be hyperbolic.

Definition 2. A linear operator $T : \mathbb{R}[x] \to \mathbb{R}[x]$ is said to *preserve hyperbolicity* (or T is a *hyperbolicity preserver*) if $T[f(x)]$ is a hyperbolic polynomial, whenever $f(x)$ is a hyperbolic polynomial.

Hyperbolicity preserving operators have been studied by virtually every author who has studied hyperbolic polynomials (see [5] and the references contained therein). The focus of our investigation involves the relationship between hyperbolicity preserving operators and hyperbolic polynomials with interlacing zeros.

Definition 3. Let $f, g \in \mathbb{R}[x]$ with $\deg(f) = n$ and $\deg(g) = m$. We say that f and g have *interlacing zeros*, if f is hyperbolic with zeros $\alpha_1, \ldots, \alpha_n$ and g is hyperbolic with zeros $\beta_1, \ldots, \beta_m$, where $|n - m| \le 1$, with one of the following forms holding:

(1) $\alpha_1 \le \beta_1 \le \alpha_2 \le \beta_2 \le \ldots \le \alpha_n \le \beta_m$,
(2) $\beta_1 \le \alpha_1 \le \beta_2 \le \alpha_2 \le \ldots \le \beta_m \le \alpha_n$,
(3) $\alpha_1 \le \beta_1 \le \alpha_2 \le \beta_2 \le \ldots \le \beta_m \le \alpha_n$, or
(4) $\beta_1 \le \alpha_1 \le \beta_2 \le \alpha_2 \le \ldots \le \alpha_n \le \beta_m$.

We will also say that the zeros of any two hyperbolic polynomials of degree 0 or 1 interlace.

Definition 4. Given two non-zero polynomials $f, g \in \mathbb{R}[x]$, we say f and g are in *proper position* and write $f \ll g$ if one of the following conditions holds:

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(1) f and g have interlacing zeros with form (1) or (4) in Definition 3 and the leading coefficients of f and g are of the same sign, or
(2) f and g have interlacing zeros with form (2) or (3) in Definition 3, and the leading coefficients of f and g are of opposite sign.

We will say that the zero polynomial is in proper position with any other hyperbolic polynomial f and write $0 \ll f$ or $f \ll 0$.

Notice that, by Definition 4, if f and g are in proper position then f and g are hyperbolic. Also, to be clear, a non-zero constant can only be in proper position with another constant or a linear polynomial. However, the zero polynomial is in proper position with any hyperbolic polynomial.

Definition 5. For any two real polynomials f and g, the *Wronskian* of f and g is defined, on $\mathbb{R}$, by

$$W[f,g] := f(x)g'(x) - f'(x)g(x).$$

It is a common exercise to show that for f and g with interlacing zeros, if $W[g,f] \leq 0$ on the whole real line then $f \ll g$.

The following Lemma demonstrates that proper position plays an important role in understanding hyperbolicity preservers.

Lemma 6 (P. Brändén [4, Lemma 2.7])**.** *Suppose the linear operator*

$$T = \sum_{k=M}^{N} Q_k(x) D^k, \tag{1.2}$$

where $Q_k(x) \in \mathbb{R}[x]$ for $M \leq k \leq N$, and $Q_M(x)Q_N(x) \not\equiv 0$, preserves hyperbolicity. Then $Q_j(x) \ll Q_{j+1}(x)$ for $M \leq j \leq N-1$. In particular, $Q_j(x)$ is hyperbolic or identically zero for all $M \leq j \leq N$.

In the special case for $M = 0$ and $N = 2$ in (1.2), we find sufficient conditions that guarantee when T preserves hyperbolicity. Our main result is the following:

Theorem 7. *Suppose Q_2, Q_1, Q_0 are real polynomials such that $deg(Q_2) = 2$, $deg(Q_1) \leq 1$, $deg(Q_0) = 0$. Then*

$$T = Q_2 D^2 + Q_1 D + Q_0$$

preserves hyperbolicity if and only if

$$W[Q_0, Q_2]^2 - W[Q_0, Q_1]W[Q_1, Q_2] \leq 0, \quad \textit{and } \; Q_0 \ll Q_1 \ll Q_2.$$

2. Quadratic Hyperbolicity Preservers

We concern ourselves with operators of the following form.

Definition 8. We will call the second order differential operators of the form

$$T = Q(x)D^2 + P(x)D + R(x) \tag{2.1}$$

a *quadratic* operator, where the polynomials, $Q(x)$ is quadratic, $P(x)$ is linear, and $R(x)$ is constant. If (2.1) is also hyperbolicity preserving, then we will refer to it as a *quadratic hyperbolicity preserver*.

The following proposition presents an operator that has been quite influential to our exposition.

Proposition 9 (Forgács et al. [1, Proposition 5])**.** If $0 < d < 1$, then the operator

$$T = (x^2 - 1)D + 2xD + d$$

preserves hyperbolicity.

For motivation, we present several other similar examples of quadratic operators.

Example 10.

$$\begin{aligned}
T_1 &= (x^2-1)D^2 + 2xD - 1 && (2.2)\\
T_2 &= (x^2-1)D^2 + 2xD + 0 && (2.3)\\
T_3 &= (x^2-1)D^2 + 2xD + 1 && (2.4)\\
T_4 &= (x^2-1)D^2 + 2xD + 2 && (2.5)\\
T_5 &= (x^2-1)D^2 - 2xD - 1 && (2.6)\\
T_6 &= (x^2-1)D^2 - 2xD + 0 && (2.7)\\
T_7 &= (x^2-1)D^2 - 2xD + 1 && (2.8)\\
T_8 &= (x^2-1)D^2 - 2xD + 2 && (2.9)
\end{aligned}$$

It was shown in [1, Lemma 5] that (2.4) is hyperbolicity preserving. Notice that $T_2 = D(x^2 - 1)D$ and thus (2.3) is hyperbolcity preserving as well. The other six examples can easily be shown to not preserve hyperbolicity.

$$\begin{aligned}
T_1[x^2-1] &= 5x^2 + 2. && (2.10)\\
T_4[(x-10)^3] &= 2(x-10)(7x^2 - 50x + 97). && (2.11)\\
T_5[x^2] &= -3x^2 - 2. && (2.12)\\
T_6[x^2] &= -2x^2 - 2. && (2.13)\\
T_7[x^2] &= -x^2 - 2. && (2.14)\\
T_8[(x-10)^3] &= 2(x-10)(x^2 + 10x + 97). && (2.15)
\end{aligned}$$

These examples show that the property of interlacing coefficients is not sufficient to establish hyperbolicity preserving. Furthermore, (2.5) demonstrates that the condition of proper position in Lemma 6 is also not sufficient to establish hyperbolicity preserving. The examples motivate us to find the necessary and sufficient conditions on the polynomial coefficients in the quadratic operator (2.1).

We will completely characterize all quadratic hyperbolicity preservers. For our characterization, we will need a result due to J. Borcea and P. Brändén.

Theorem 11 (J. Borcea, P. Brändén [3, Theorem 1.3])**.** *Let $T : \mathbb{R}[x] \to \mathbb{R}[x]$ be a finite differential linear operator, thus there exists real polynomials $\{Q_k(x)\}_{k=0}^n$ such that*

$$T = \sum_{k=0}^{n} Q_k(x) D^k.$$

T is hyperbolicity preserving, if and only if,

$$\sum_{k=0}^{n} Q_k(x)(-w)^k \neq 0.$$

for every $x, w \in H^+$.

In general, Theorem 11 can be difficult to apply since very little is known about two variable *stable* polynomials (See [3]). The next few lemmas establish a small class of two variable stable polynomials.

Lemma 12. *Let* $A,\ B \in \mathbb{C} - \mathbb{R}$ *be two non-real complex numbers such that*

(i) $0 < \mathrm{Arg}(B) < \mathrm{Arg}(A) < 2\pi$,
(ii) $\mathrm{Arg}(A) - \mathrm{Arg}(B) < \pi$, and
(iii) $\mathrm{Im}(A) < \mathrm{Im}(B)$.

Then for any $r_1,\ r_2 \in \mathbb{R}$, $r_1 < r_2$, *there is* $x, w \in H^+$ *such that* $(x + r_1)w = A$ *and* $(x + r_2)w = B$.

Proof. Consider the following cases.

Case 1: $B \in H^+$. The point B may be located in either quadrant I, on the imaginary axis, or in quadrant II, as described in Figure 2.1. The hypotheses (i), (ii), and (iii) implies that point A is located somewhere in the shaded region of the corresponding point B. Define the function $f : [0, \mathrm{Arg}(B)] \to \mathbb{R}$ by

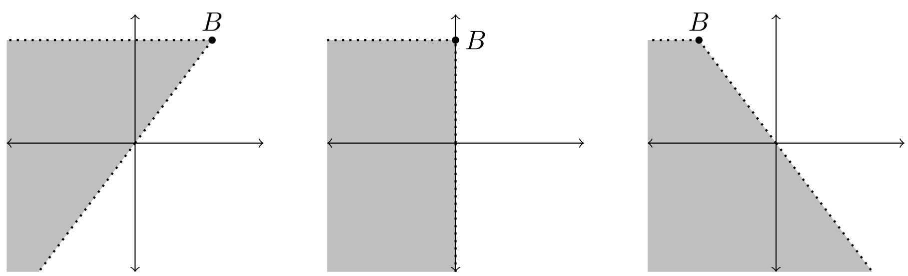


FIGURE 2.1

$$f(\theta) := \mathrm{Im}(e^{-i\theta}A) - \mathrm{Im}(e^{-i\theta}B). \tag{2.16}$$

Then $f(0) < 0$ by (iii), and $f(\mathrm{Arg}(B)) > 0$ by (ii). Thus by continuity, there exist $\theta_0 \in (0, \mathrm{Arg}(B))$ such that $f(\theta_0) = 0$, which implies that $(e^{-i\theta_0}B - e^{-i\theta_0}A) > 0$ by (i). Define the function $g : [0, \infty) \to \mathbb{R}$ by

$$g(k) := k(e^{-i\theta_0}B - e^{-i\theta_0}A). \tag{2.17}$$

Notice $g \geq 0$, $g(0) = 0$, and $\lim_{k\to+\infty} g(k) = +\infty$. Thus, there exist $k_0 > 0$ such that $g(k_0) = r_2 - r_1$. Let

$$x = \frac{1}{2}(k_0e^{-i\theta_0}B + k_0e^{-i\theta_0}A - r_1 - r_2), \text{ and } w = \frac{1}{k_0}e^{i\theta_0}. \tag{2.18}$$

It follows that $x, w \in H^+$, $(x + r_1)w = A$, and $(x + r_2)w = B$.

Case 2: $B \in H^-$. Similar to Case 1, the point B may be located in either quadrant III, on the imaginary axis, or in quadrant IV, as described in Figure (2.2). Point A is located somewhere in the shaded region of the corresponding point B by hypotheses (i), (ii), and (iii).

Define the function $f : [0, 2\pi - \mathrm{Arg}(B)] \to \mathbb{R}$ by

$$f(\theta) := \mathrm{Im}(e^{i\theta}A) - \mathrm{Im}(e^{i\theta}B). \tag{2.19}$$

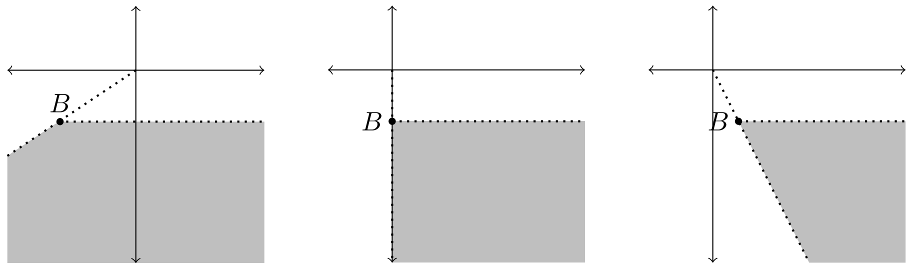


FIGURE 2.2

Then $f(0) < 0$ by (iii), and $f(2\pi - \text{Arg}(B)) > 0$ by (ii). Thus by continuity, there exist $\theta_0 \in (0, 2\pi - \text{Arg}(B))$ such that $f(\theta_0) = 0$, which implies that $(e^{i\theta_0}B - e^{i\theta_0}A) < 0$ by (i). Define the function $g : (-\infty, 0] \to \mathbb{R}$ by

$$g(k) := k(e^{i\theta_0}B - e^{i\theta_0}A). \tag{2.20}$$

Then $g \geq 0$, $g(0) = 0$, and $\lim\limits_{k\to-\infty} g(k) = +\infty$. Thus, there exist $k_0 < 0$ such that $g(k_0) = r_2 - r_1$. Let

$$x = \frac{1}{2}(k_0 e^{i\theta_0}B + k_0 e^{i\theta_0}A - r_1 - r_2), \text{ and } w = \frac{1}{k_0}e^{-i\theta_0}. \tag{2.21}$$

It follows that $x, w \in H^+$, $(x + r_1)w = A$, and $(x + r_2)w = B$. □

Lemma 13. *Let $a, b, r_1, r_2, r \in \mathbb{R}$, $a, b \geq 0$, and $r_1 \neq r_2$. Set*

$$f(x, w) = ((x + r_1)w - a)((x + r_2)w - b), \quad x, w \in \mathbb{C}. \tag{2.22}$$

Then

$$f(x, w) \neq r \quad \forall\, x, w \in H^+ \quad \textit{if and only if} \quad r \in [0, ab].$$

Proof. Since the factors of $f(x, w)$ in (2.22) are symmetric, we let $r_1 < r_2$. There are three cases to prove necessity. The following is the outline.

Case 1. $r \in (-\infty, 0)$, and $a < b + 2\sqrt{|r|}$.

Case 2. $r \in (-\infty, 0)$, and $a \geq b + 2\sqrt{|r|}$.

Case 3. $r \in (ab, \infty)$.

We show in each case that there exist $x, w \in H^+$ such that $f(x, w) = r$.

Case 1. Consider $r \in (-\infty, 0)$, and $a < b + 2\sqrt{|r|}$. Define $g : [0, \pi/2] \to \mathbb{R}$ by

$$\begin{aligned} g(\theta) :=& \left(\sqrt{|r|}e^{i\theta} + b\right) - \left(\sqrt{|r|}e^{i(\pi-\theta)} + a\right) \\ =& \sqrt{|r|}(2\cos(\theta)) - a + b. \end{aligned} \tag{2.23}$$

The function g is real valued and $g(0) = b + 2\sqrt{|r|} - a > 0$ by assumption. Thus by continuity, there exists $\theta_0 \in (0, \pi/2)$ such that $g(\theta_0) > 0$, which implies the following.

(a) $\text{Im}\left(\sqrt{|r|}e^{i\theta_0} + b\right) - \text{Im}\left(\sqrt{|r|}e^{i(\pi-\theta_0)} + a\right) = 0$,

(b) $\text{Re}\left(\sqrt{|r|}e^{i\theta_0} + b\right) - \text{Re}\left(\sqrt{|r|}e^{i(\pi-\theta_0)} + a\right) > 0$, and

(c) $\left(\sqrt{|r|}e^{i\theta_0} + b\right), \left(\sqrt{|r|}e^{i(\pi-\theta_0)} + a\right) \in H^+$.

By (a), (b), and (c),

$$\operatorname{Arg}\left(\sqrt{|r|}e^{i(\pi-\theta_0)}+a\right)-\operatorname{Arg}\left(\sqrt{|r|}e^{i\theta_0}+b\right)>0. \tag{2.24}$$

Define the function $h:(0,1]\to\mathbb{R}$ by

$$h(k):=\operatorname{Arg}\left(k\sqrt{|r|}e^{i(\pi-\theta_0)}+a\right)-\operatorname{Arg}\left(\frac{\sqrt{|r|}}{k}e^{i\theta_0}+b\right). \tag{2.25}$$

The function h is real valued, and $h(1)>0$. Thus by continuity, there exists $k_0<1$ such that

$$\operatorname{Arg}\left(k_0\sqrt{|r|}e^{i(\pi-\theta_0)}+a\right)-\operatorname{Arg}\left(\frac{\sqrt{|r|}}{k_0}e^{i\theta_0}+b\right)>0, \tag{2.26}$$

such that

$$\operatorname{Im}\left(k_0\sqrt{|r|}e^{i(\pi-\theta_0)}+a\right)<\operatorname{Im}\left(\frac{\sqrt{|r|}}{k_0}e^{i\theta_0}+b\right). \tag{2.27}$$

Let

$$A=k_0\sqrt{|r|}e^{i(\pi-\theta_0)}+a, \quad \text{and} \quad B=\frac{\sqrt{|r|}}{k_0}e^{i\theta_0}+b. \tag{2.28}$$

Then (2.26) and (2.27) satisfies items (i), (ii), and (iii) of Lemma 12, hence there exist $x,w\in H^+$ such that $(x+r_1)w=A$ and $(x+r_2)w=B$. Thus,

$$\begin{aligned} f(x,w) &= ((x+r_1)w-a)((x+r_2)w-b)\\ &= \left(k_0\sqrt{|r|}e^{i(\pi-\theta_0)}\right)\left(\frac{\sqrt{|r|}}{k_0}e^{i\theta_0}\right)=-|r|=r. \end{aligned} \tag{2.29}$$

Case 2: We consider $r\in(-\infty,0)$, and $b+2\sqrt{|r|}\le a$. We will only need $b<a+2\sqrt{|r|}$. This is easily seen to be true by adding $2\sqrt{|r|}$ to both sides of $b+2\sqrt{|r|}\le a$, and observing $b<b+4\sqrt{|r|}$. Define the function $g:[0,\pi/2]\to\mathbb{R}$ by

$$\begin{aligned} g(\theta) &:= \left(\sqrt{|r|}e^{i(2\pi-\theta)}+a\right)-\left(\sqrt{|r|}e^{i(\pi+\theta)}+b\right)\\ &= \sqrt{|r|}(2\cos(\theta))+a-b. \end{aligned} \tag{2.30}$$

Again, g is real valued, and $g(0)=a+2\sqrt{|r|}-b>0$. Thus by continuity, there exists $\theta_0\in(0,\pi/2)$ such that $g(\theta_0)>0$, which implies the following:

(a) $\operatorname{Im}\left(\sqrt{|r|}e^{i(2\pi-\theta_0)}+a\right)-\operatorname{Im}\left(\sqrt{|r|}e^{i(\pi+\theta_0)}+b\right)=0$,
(b) $\operatorname{Re}\left(\sqrt{|r|}e^{i(2\pi-\theta_0)}+a\right)-\operatorname{Re}\left(\sqrt{|r|}e^{i(\pi+\theta_0)}+b\right)>0$,
(c) $\left(\sqrt{|r|}e^{i(2\pi-\theta_0)}+a\right),\ \left(\sqrt{|r|}e^{i(\pi+\theta_0)}+b\right)\in H^-$.

By (a), (b), and (c),

$$\operatorname{Arg}\left(\sqrt{|r|}e^{i(2\pi-\theta_0)}+a\right)-\operatorname{Arg}\left(\sqrt{|r|}e^{i(\pi+\theta_0)}+b\right)>0. \tag{2.31}$$

Define the function $h:[1,\infty)\to\mathbb{R}$ by

$$h(k):=\operatorname{Arg}\left(k\sqrt{|r|}e^{i(2\pi-\theta_0)}+a\right)-\operatorname{Arg}\left(\frac{\sqrt{|r|}}{k}e^{i(\pi+\theta_0)}+b\right). \tag{2.32}$$

The function h is real valued, and $h(1) > 0$. Thus by continuity, there exists $k_0 > 1$ such that

$$\operatorname{Arg}\left(k_0\sqrt{|r|}e^{i(2\pi-\theta_0)}+a\right)-\operatorname{Arg}\left(\frac{\sqrt{|r|}}{k_0}e^{i(\pi+\theta_0)}+b\right)>0, \tag{2.33}$$

so that

$$\operatorname{Im}\left(k_0\sqrt{|r|}e^{i(2\pi-\theta_0)}+a\right)<\operatorname{Im}\left(\frac{\sqrt{|r|}}{k_0}e^{i(\pi+\theta_0)}+b\right). \tag{2.34}$$

Let

$$A=k_0\sqrt{|r|}e^{i(2\pi-\theta_0)}+a, \quad \text{and} \quad B=\frac{\sqrt{|r|}}{k_0}e^{i(\pi+\theta_0)}+b. \tag{2.35}$$

Then (2.33) and (2.34) satisfies items (i), (ii), and (iii) of Lemma 12, hence there exist $x, w \in H^+$ such that $(x+r_1)w = A$ and $(x+r_2)w = B$. Thus,

$$\begin{aligned} f(x,w) &= ((x+r_1)w-a)((x+r_2)w-b) \\ &= \left(k_0\sqrt{|r|}e^{i(2\pi-\theta_0)}\right)\left(\frac{\sqrt{|r|}}{k_0}e^{i(\pi+\theta_0)}\right)=-|r|=r. \end{aligned} \tag{2.36}$$

Case 3: We consider $r \in (ab, \infty)$. Since $r > ab$, $r = a'b'$, for some $a' > a$, and $b' > b$. Define the function $g : [\pi/2, \pi] \to [a - a', a] \times [b - b', b]$ by

$$g(\theta) := \left(\operatorname{Re}(a'e^{-i\theta})+a,\, \operatorname{Re}(b'e^{i\theta})+b\right). \tag{2.37}$$

Since $a-a', b-b' < 0$, $g(\pi) = (a-a', b-b')$ has negative coordinates. By continuity, there exists $\theta_0 \in (\pi/2, \pi)$ such that $g(\theta_0)$ has negative coordinates, which implies that $a'e^{-i\theta_0} + a$ is in quadrant three, and $b'e^{i\theta_0} + b$ is in quadrant two. Let

$$A = a'e^{-i\theta_0}+a, \quad \text{and} \quad B = b'e^{i\theta_0}+b. \tag{2.38}$$

Again, by Lemma 12, there exist $x, w \in H^+$ such that $(x + r_1)w = A$, and $(x + r_2)w = B$. Thus,

$$f(x,w) = ((x+r_1)w-a)((x+r_2)w-b) = \left(a'e^{-\theta_0 i}\right)\left(b'e^{\theta_0 i}\right) = a'b' = r. \tag{2.39}$$

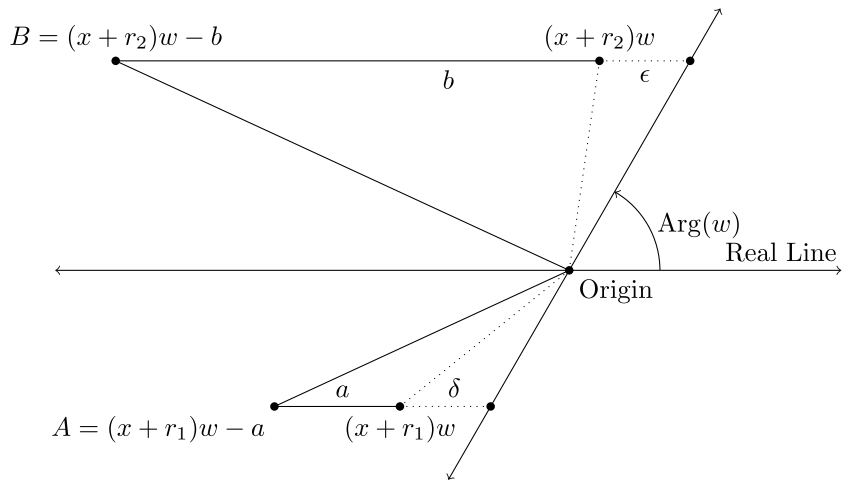
To prove sufficiency, first consider $r \in (0, ab]$. By way of contradiction, assume there exist $x, w \in H^+$ such that $((x + r_1)w - a)((x + r_2)w - b) = r$. Let $A = ((x + r_1)w - a)$, $B = ((x + r_2)w - b)$. Since $x + r_1, x + r_2 \in H^+$, the rotation by $\operatorname{Arg}(w) \in (0, \pi)$ and the shifts to the left by $a, b > 0$ restrict the location of A and B considerably. Indeed, since AB is a positive real number, $\operatorname{Arg}(A) + \operatorname{Arg}(B) = 0 \pmod{2\pi}$. In particular, as $r_1 < r_2$, B must be in H^+, which implies

$$0 < \operatorname{Arg}(w) < \operatorname{Arg}((x+r_2)w) < \operatorname{Arg}((x+r_2)w-b) < \pi, \tag{2.40}$$

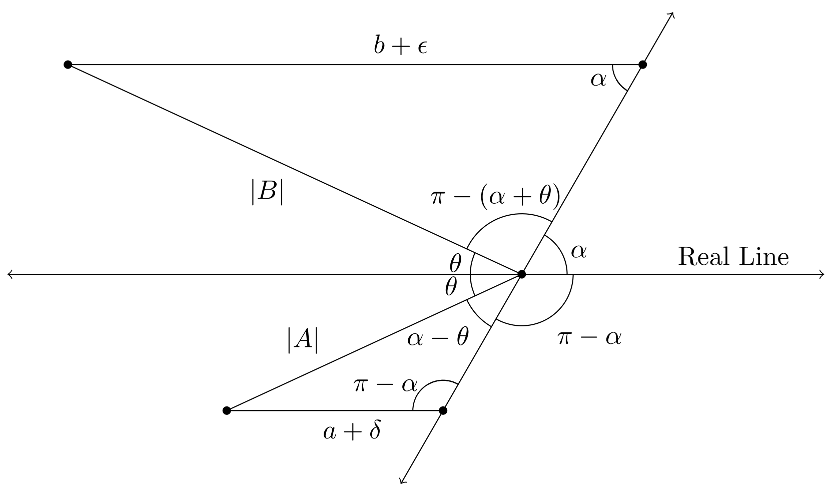
and A must be in H^-, which implies

$$\pi < \operatorname{Arg}((x+r_1)w-a) < \operatorname{Arg}((x+r_1)w) < \pi-\operatorname{Arg}(w) < 2\pi. \tag{2.41}$$

The following figure illustrates inequalities (2.40) and (2.41).



We let ϵ and δ be the horizontal distance from $(x + r_1)w$ and $(x + r_2)w$ to the line formed by Arg(w). In fact, $\delta = \frac{\text{Im}(x+r_1)}{\sin(Arg(w))}$, and $\epsilon = \frac{\text{Im}(x+r_2)}{\sin(Arg(w))}$, so that $\delta = \epsilon > 0$. We redraw the picture with different labels and examine the points geometrically.



The inequalities $\alpha - \theta > 0$ and $\pi - (\alpha + \theta) > 0$ imply $0 < \theta < \alpha < \pi - \theta < \pi$, so that

$$\sin(\theta) < \sin(\alpha),$$

since $\sin(\theta) = \sin(\pi - \theta)$. Thus,

$$0 < \left(\frac{\sin(\theta)}{\sin(\alpha)}\right)^2 < 1, \tag{2.42}$$

and the law of sines yield that

$$\begin{aligned} (a+\delta)(b+\epsilon) &= \frac{|A|\sin(\alpha-\theta)}{\sin(\pi-\alpha)} \cdot \frac{|B|\sin(\pi-(\alpha+\theta))}{\sin(\alpha)} \\ &= \left(1 - \left(\frac{\sin(\theta)}{\sin(\alpha)}\right)^2\right)|AB| < |AB|. \end{aligned} \tag{2.43}$$

Hence we have the contradiction that

$$ab < (a+\delta)(b+\epsilon) < |AB| = r. \tag{2.44}$$

To finish the proof, consider $r = 0$. By way of contradiction, suppose there are $x, w \in H^+$ such that

$$((x+r_1)w - a)((x+r_2)w - b) = 0.$$

Thus, $(x+r_1)w = a$, or $(x+r_2)w = b$. However, neither of these can hold, since the product of any two complex numbers in H^+ cannot be a non negative real number. ☐

Theorem 14. *Let $a, b \geq 0$, $r_1, r_2, R \in \mathbb{R}$, and $r_1 \neq r_2$. Then,*

$$R \in [0, ab]$$

if and only if

$$T := (x+r_1)(x+r_2)D^2 + (b(x+r_1) + a(x+r_2))\, D + R$$

(where $D := \frac{d}{dx}$), is hyperbolicity preserving.

Proof. ($\Rightarrow$) Assume $R \in [0, ab]$. By Theorem 11, it suffices to show for every $x, w \in H^+$,

$$(x+r_1)(x+r_2)w^2 - (b(x+r_1) + a(x+r_2))\, w + R \neq 0. \tag{2.45}$$

We assume on the contrary that (2.45) is false for some $x, w \in H^+$. We factor (2.45) to attain

$$((x+r_1)w - a)((x+r_2)w - b) = ab - R, \tag{2.46}$$

which is impossible by Lemma 13, a contradiction.

($\Leftarrow$) Suppose T is hyperbolicity preserving. By Theorem 11, for every $x, w \in H^+$,

$$(x+r_1)(x+r_2)w^2 - (b(x+r_1) + a(x+r_2))\, w + R \neq 0. \tag{2.47}$$

We factor (2.47) to attain

$$((x+r_1)w - a)((x+r_2)w - b) \neq ab - R, \quad \forall\, x, w \in H^+, \tag{2.48}$$

which implies that $R \in [0, ab]$ by Lemma 13. ☐

Theorem 15. *For $c_i, r_j \in \mathbb{R}$, $i = 0,1,2$, $j = 1,2,3$, $c_2 \neq 0$, $r_1 \neq r_2$, let $Q_0(x) = c_0$, $Q_1(x) = c_1(x - r_3)$, $Q_2(x) = c_2(x-r_1)(x-r_2)$. Then*

$$0 \leq c_1^2 \left(\frac{(r_1 - r_3)(r_3 - r_2)}{(r_2 - r_1)^2} \right) - c_0 c_2,$$

and c_0, c_1, c_2 are of the same sign if and only if

$$T := Q_2(x)D^2 + Q_1(x)D + Q_0(x)$$

(where $D := \frac{d}{dx}$), preserves hyperbolicity.

Proof. To prove sufficiency, if T preserves hyperbolicity, then by Lemma 6, c_i, $i = 0,1,2$ are of the same sign and the zeros of Q_2 and Q_1 interlace. Since

$$T = c_2 \left((x-r_1)(x-r_2)D^2 + \frac{c_1}{c_2}(x - r_3)D + \frac{c_0}{c_2} \right) \tag{2.49}$$

$$= c_2 \Bigg((x-r_1)(x-r_2)D^2$$

$$+ \frac{c_1}{c_2}\left[\frac{(r_1-r_3)}{(r_1-r_2)}(x-r_2) + \frac{(r_3-r_2)}{(r_1-r_2)}(x-r_1)\right]D + \frac{c_0}{c_2}\Bigg), \tag{2.50}$$

then by Theorem 14,

$$\frac{c_0}{c_2} \in \left[0,\ \left(\frac{c_1}{c_2}\right)^2 \frac{(r_1-r_3)(r_3-r_2)}{(r_1-r_2)^2}\right], \tag{2.51}$$

and so

$$0 \le c_1^2\left(\frac{(r_1-r_3)(r_3-r_2)}{(r_2-r_1)^2}\right) - c_0c_2. \tag{2.52}$$

To prove necessity, suppose c_i, $i = 0, 1, 2$ are of the same sign, and

$$0 \le c_1^2\left(\frac{(r_1-r_3)(r_3-r_2)}{(r_2-r_1)^2}\right) - c_0c_2. \tag{2.53}$$

We want to conclude that

$$\frac{c_1}{c_2}\frac{(r_1-r_3)}{(r_1-r_2)}, \frac{c_1}{c_2}\frac{(r_3-r_2)}{(r_1-r_2)} \ge 0. \tag{2.54}$$

To this end, if $c_1 = 0$, then (2.54) holds immediately. Suppose $c_1 \neq 0$, and that $r_1 < r_2$. Then (2.53) implies $0 \le (r_1 - r_3)(r_3 - r_2)$, and we conclude that $r_1 \le r_3 \le r_2$ (i.e., $r_3 < r_1 < r_2$ cannot hold, since it implies $(r_1 - r_3)(r_3 - r_2) < 0$, and also if $r_1 < r_2 < r_3$, then $(r_1 - r_3)(r_3 - r_2) < 0$), and hence, (2.54) holds. By symmetry, the same conclusion is true if $r_2 < r_1$. Thus by Theorem 14, T preserves hyperbolicity. □

The equality of (2.49) and (2.50) uses a fact established in Fisk's polynomial book [6, p. 13, Lemma 1.20], although because our case is easy to verify, we do not need its full strength. For the sake of completeness, we state the result seen in Fisk's book.

Lemma 16 (Fisk [6, p. 13, Lemma 1.20])**.** *Assume that f is a polynomial of degree n, with positive leading coefficient, and with real zeros $\{a_1, \dots, a_n\}$. Suppose that g is a polynomial with positive leading coefficient. If g has degree $n-1$, and we write*

$$g(x) = c_1\frac{f(x)}{x-a_1} + \ldots + c_n\frac{f(x)}{x-a_n},$$

then f and g have interlacing zeros if and only if all $c_i \ge 0$ for $i = 1, 2, \dots, n$.

We now remove the condition of Q_2 having distinct zeros. We begin with a lemma that is analogous to Lemma 13.

Lemma 17. *Let $a, r \in \mathbb{R}$, $a \ge 0$. Set*

$$f(x) := x^2 - ax + r, \quad x \in \mathbb{C}.$$

Then

$$f(x) \neq 0 \ \ \forall\, x \in \mathbb{C} - [0,\infty) \quad \textit{if and only if} \quad r \in \left[0, \frac{a^2}{4}\right].$$

Proof. The zeros of f are $\frac{1}{2}\left(a \pm \sqrt{a^2 - 4r}\right)$. There are two cases to prove necessity.

Case 1. If $r < 0$, then one of the zeros of f is a negative real number, thus there exist $x \in \mathbb{C} - [0,\infty)$ such that $f(x) = 0$.

Case 2. If $r > a^2/4$, then f has two imaginary zeros, thus the zeros of f are in $\mathbb{C} - [0,\infty)$.

To prove sufficiency, suppose $0 \le r \le a^2/4$. Then f has two non-negative zeros, so that f never vanishes in $\mathbb{C} - [0, \infty)$. □

Theorem 18. *Let $a \ge 0$, $r, R \in \mathbb{R}$. Then,*

$$R \in \left[0, \frac{a^2}{4}\right]$$

if and only if

$$T := (x + r)^2 D^2 + a(x + r)D + R$$

is hyperbolicity preserving.

Proof. ($\Rightarrow$) Assume $R \in [0, a^2/4]$. By Theorem 11, it suffices to show for every $x, w \in H^+$,

$$(x + r)^2 w^2 - a(x + r)w + R \ne 0. \tag{2.55}$$

We assume on the contrary that (2.55) is false for some $x, w \in H^+$. Let $z = (x+r)w$ in (2.55), so that $z \in \mathbb{C} - [0, \infty)$, and

$$z^2 - az + R = 0. \tag{2.56}$$

This is impossible by Lemma 17, a contradiction.

($\Leftarrow$) Suppose T is hyperbolicity preserving. By Theorem 11, for every $x, w \in H^+$,

$$(x + r)^2 w^2 - a(x + r)w + R \ne 0. \tag{2.57}$$

Let $z = (x + r)w$ in (2.57), so that $z \in \mathbb{C} - [0, \infty)$, and

$$z^2 - az + R \ne 0, \quad \forall\, z \in \mathbb{C} - [0, \infty), \tag{2.58}$$

which implies that $R \in [0, a^2/4]$ by Lemma 17. □

The analogous statement of Theorem 15 is the following, and its proof follows *mutatis mutandis*, from the proof of Theorem 15.

Theorem 19. *For $r, c_i \in \mathbb{R}$, $i = 0, 1, 2$, $c_2 \ne 0$, let $Q_0(x) = c_0$, $Q_1(x) = c_1(x - r)$, $Q_2(x) = c_2(x - r)^2$. Then*

$$0 \le c_1^2 \left(\frac{1}{4}\right) - c_0 c_2$$

and c_0, c_1, c_2 are of the same sign, if and only if

$$T = Q_2(x)D^2 + Q_1(x)D + Q_0(x)$$

preserves hyperbolicity.

We now wish to find a condition that combines the statements of Theorem 19 and Theorem 15. But first a Lemma.

Lemma 20. *For $c_i, r_j \in \mathbb{R}$, $i = 0, 1, 2$, $j = 1, 2, 3$, $c_2 \ne 0$, let $Q_0(x) = c_0$, $Q_1(x) = c_1(x - r_3)$, $Q_2(x) = c_2(x - r_1)(x - r_2)$. If $T = Q_2(x)D^2 + Q_1(x)D + Q_0(x)$ is hyperbolicity preserving then*

$$0 \le c_1^2 - 4c_0c_2.$$

Furthermore, if $r_1 \ne r_2$ then

$$0 \le c_1^2 \frac{(r_1 - r_3)(r_3 - r_2)}{(r_2 - r_1)^2} - c_0 c_2 \le c_1^2 \frac{1}{4} - c_0 c_2.$$

Thus, if $0 = c_1^2 - 4c_0c_2$ then $2r_3 = r_1 + r_2$.

Proof. Theorem 19 deals with the case of when $r_1 = r_2$, thus it suffices to show

$$0 \le \frac{(r_1 - r_3)(r_3 - r_2)}{(r_2 - r_1)^2} \le \frac{1}{4}. \tag{2.59}$$

The left inequality holds because Q_2 and Q_1 have interlacing zeros by Lemma 6. To show the right inequality we proceed as follows,

$$0 \le (2r_3 - (r_1 + r_2))^2, \tag{2.60}$$

$$4(r_1r_3 + r_2r_3) \le (r_2 + r_1)^2 + 4r_3^2, \tag{2.61}$$

$$4(r_1r_3 - r_1r_2 - r_3^2 + r_2r_3) \le r_2^2 - 2r_1r_2 + r_1^2, \tag{2.62}$$

$$4(r_1 - r_3)(r_3 - r_2) \le (r_2 - r_1)^2. \tag{2.63}$$

□

Theorem 21. *For $c_i, r_j \in \mathbb{R}$, $i = 0, 1, 2$, $j = 1, 2, 3$, $c_2 \ne 0$, let $Q_0(x) = c_0$, $Q_1(x) = c_1(x - r_3)$, $Q_2(x) = c_2(x - r_1)(x - r_2)$ with $Q_0(x) \ll Q_1(x) \ll Q_2(x)$. Then*

$$T = Q_2(x)D^2 + Q_1(x)D + Q_0(x)$$

preserves hyperbolicty if and only if

$$W[Q_0, Q_2]^2 - W[Q_0, Q_1]W[Q_1, Q_2] \le 0.$$

Proof. Since we are assuming $Q_0 \ll Q_1 \ll Q_2$ then c_0, c_1, c_2 are of the same sign and $r_1 \le r_3 \le r_2$. Define,

$$\begin{aligned} w(x) \quad &:= \quad W[Q_0, Q_2]^2 - W[Q_0, Q_1]W[Q_1, Q_2] \\ &= \quad c_0c_2(4c_0c_2 - c_1^2)x^2 + 2c_0c_2(-2c_0c_2(r_1 + r_2) + c_1^2r_3)x \\ &\quad\quad + c_0c_2(c_0c_2(r_1 + r_2)^2 + c_1^2(r_1r_2 - r_1r_3 - r_2r_3)). \end{aligned}$$

Suppose $r_1 = r_2$, then $w(x) = -c_0c_2(c_1^2 - 4c_0c_2)(x - r_1)^2$. It is clear that $w(x) \le 0$ if and only if $0 \le c_1^2 - 4c_0c_2$, thus we apply Theorem 19.

Suppose $0 = c_1^2 - 4c_0c_2$ and $r_1 \ne r_2$. By Lemma 20, Theorem 15 can restated as, "T is hyperbolicity preserving if and only if $2r_3 = r_1 + r_2$". We recalculate w, under the assumption that $0 = c_1^2 - 4c_0c_2$,

$$w(x) = 4c_0^2c_2^2(2r_3 - r_1 - r_2)x + c_0^2c_2^2(2(r_1 + r_2)(r_1 + r_2 - 2r_3) - (r_1 - r_2)^2). \tag{2.64}$$

We now see that, $w(x) \le 0$, if and only if, $2r_3 = r_1 + r_2$.

Thus we may assume $0 \ne c_1^2 - 4c_0c_2$ and $r_1 \ne r_2$, in which case w is a quadratic with vertex

$$\left(r_3, \frac{c_0c_1^2c_2}{c_1^2 - 4c_0c_2} \left(c_0c_2(r_1 - r_2)^2 + c_1^2(r_1 - r_3)(r_2 - r_3) \right) \right). \tag{2.65}$$

Since w is a quadratic then $w(x) \le 0$ if and only if the leading coefficient

$$c_0c_2(4c_0c_1 - c_1^2) < 0 \tag{2.66}$$

and y-coordinate of the vertex

$$\frac{c_0c_1^2c_2}{c_1^2 - 4c_0c_2} \left(c_0c_2(r_1 - r_2)^2 + c_1^2(r_1 - r_3)(r_2 - r_3) \right) \le 0. \tag{2.67}$$

Thus, we can say that $w(x) \le 0$ if and only if $0 < c_1^2 - 4c_0c_1$ and $0 \le c_1^2(r_1 - r_3)(r_3 - r_2) - c_0c_2(r_1 - r_2)^2$. By Lemma 20 and Theorem 15 those conditions are equivalent to T preserving hyperbolicity. □

It is unnecessary to assume that the polynomial coefficients of T have real zeros, as this will natural follow from 6. Furthermore if Q_2 is a quadratic then Lemma 6 states that Q_1 cannot be a non-zero constant if T is to preserve hyperbolicity. To summarize we restate Theorem 21 with a little more generality.

Theorem 7. Suppose Q_2, Q_1, Q_0 are real polynomials such that $deg(Q_2) = 2$, $deg(Q_1) \leq 1$, $deg(Q_0) = 0$. Then

$$T = Q_2 D^2 + Q_1 D + Q_0$$

preserves hyperbolicty if and only if

$$W[Q_0, Q_2]^2 - W[Q_0, Q_1]W[Q_1, Q_2] \leq 0, \text{ and } Q_0 \ll Q_1 \ll Q_2.$$

3. Multiplier Sequences

We now wish to establish several consequences of the above quadratic operators.

Definition 22. Let $\{P_n\}$ be a basis for $\mathbb{R}[x]$. Let $\{A_n\}$ be a sequence of real numbers. If there is a linear operator, T, such that $T[P_n] = A_n P_n$ for every $n \in \mathbb{N}$, then we call T a *P_n-multiplier operator*. We will sometimes write $T = \{A_n\}$ when there is no question of the basis. Likewise, if there is a hyperbolicity preserver, T, such that $T[P_n] = A_n P_n$ for every $n \in \mathbb{N}$, then we call T a *P_n-multiplier squence* and sometimes write $T = \{A_n\}$.

There is a natural relationship between differential equations, differential operators, and P_n-multiplier sequences. This relationship has been used ([1, 7]), but never explicitly stated.

Theorem 23. *Let P_n be a basis for $\mathbb{R}[x]$. Suppose for each $n \in \mathbb{N}$, P_n satisfies the differential equation*

$$\sum_{k=0}^{\infty} Q_k(x) y^{(k)} = A_n y$$

where $\{Q_k\}$ is a sequence of real polynomials and $\{A_n\}$ is a sequence of real numbers. Then A_n is a P_n-multiplier sequence if and only if

$$\sum_{k=0}^{\infty} Q_k D^k$$

is a hyperbolicity preserver.

Using Theorem 15 we can restate the above theorem.

Theorem 24. *Let P_n be a simple set for $\mathbb{R}[x]$ and let $\{A_n\}$ be a sequence of real numbers. Let $c_i, r_j \in \mathbb{R}$, $i = 0, 1, 2$, $j = 1, 2, 3$, $c_2 \neq 0$. Suppose for each $n \in \mathbb{N}$ that P_n satisfies the differential equation,*

$$c_2(x - r_1)(x - r_2)y'' + c_1(x - r_3)y' + c_0 y = A_n y$$

then $\{A_n\}$ is a P_n-multiplier sequence, if and only if, c_0, c_1, c_2 are of the same sign and

$$0 \leq c_1^2 \left(\frac{(r_1 - r_3)(r_3 - r_2)}{(r_2 - r_1)^2} \right) - c_2 c_0.$$

In light of Theorem 19, we take $\frac{(r_1 - r_3)(r_3 - r_2)}{(r_2 - r_1)^2} = \frac{1}{4}$ in the case that $r_1 = r_2$.

A large number of very well known bases for $\mathbb{R}[x]$ satisfy differential equations of the above form ([12, pg. 173, 188, 204, 258]). We exhibit classes of multiplier sequences for Legendre, Jacobi, and the standard basis. We state the cooresponding differential equations:

Standard Basis

$$Ax^2(x^n)'' + Bx(x^n)' + C(x^n) = (An(n-1) + Bn + C)x^n \tag{3.1}$$

Lengendre Polynomials

$$A(x^2-1)P_n'' + 2AxP_n' + BP_n = (An(n+1) + B)P_n \tag{3.2}$$

Jacobi Polynomials

$$\begin{aligned} A(x^2-1)(P_n^{\alpha,\beta})'' + A((\alpha+\beta+2)x - (\beta-\alpha))(P_n^{\alpha,\beta})' + BP_n^{\alpha,\beta} \\ = (An(n+\alpha+\beta+1) + B)P_n^{\alpha,\beta} \end{aligned} \tag{3.3}$$

We now establish several classes of multiplier sequences.

Theorem 25. *Let $A, B, C \in \mathbb{R}$. Then $\{An(n-1)+Bn+C\}$ is a classic multiplier sequence if and only if A, B, C are of the same sign and*

$$0 \leq B^2 - 4AC.$$

Theorem 26. *Let $A, B \in \mathbb{R}$, $A \neq 0$. Then $\{An(n+1)+B\}$ is a P_n-multiplier sequence (Lengendre multiplier sequence) if and only if*

$$0 \leq \frac{B}{A} \leq 1.$$

Theorem 27. *Let $A, B \in \mathbb{R}$, $A \neq 0$. Then $\{An(n+\alpha+\beta+1)+B\}$ is a $P_n^{\alpha,\beta}$-multiplier sequence (Jacobi multiplier sequence) if and only if*

$$0 \leq \frac{B}{A} \leq (\alpha+1)(\beta+1) \text{ and } -1 \leq \alpha, \beta.$$

Proof. Notice,

$$(A(\alpha+\beta+2))^2\left(1 - \frac{\beta-\alpha}{\alpha+\beta+2}\right)\left(1 + \frac{\beta-\alpha}{\alpha+\beta+2}\right) - 4AB \geq 0 \tag{3.4}$$

and

$$A, A(\alpha+\beta+2), B \text{ are of the same sign} \tag{3.5}$$

is equivalent to

$$0 \leq \frac{B}{A} \leq (\alpha+1)(\beta+1) \text{ and } -1 \leq \alpha, \beta. \tag{3.6}$$

□

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